AN EXPLICIT FORMULA FOR THE LINEARIZATION COEFFICIENTS OF BESSEL POLYNOMIALS II

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Abstract. In this paper, a single sum formula for the linearization coefficients of the Bessel polynomials is given. In three special cases this formula reduces indeed to either Atia and Zeng's formula (Ramanujan Journal, Doi 10.1007/s11139-011-9348-4) or Berg and Vignat's formulas in their proof of the positivity results about these coefficients (Constructive Approximation, **27** (2008), 15-32). As a bonus, a formula reducing a sum of hypergeometric functions ${}_3F_2$ to ${}_2F_1$ is obtained.

Keywords Bessel polynomials, Linearization coefficients.

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1. Introduction

The Bessel polynomials q_n of degree n are defined by

$$q_n(u) = \sum_{k=0}^{n} \frac{(-n)_k 2^k}{(-2n)_k k!} u^k, \tag{1}$$

where we use the Pochhammer symbol $(z)_n := z(z+1)\dots(z+n-1)$ for $z \in \mathbb{C}$, $n = 0, 1, \dots$. The first values are

$$q_0(u) = 1, \quad q_1(u) = 1 + u, \quad q_2(u) = 1 + u + \frac{u^2}{3}.$$

Some recursion formulas for q_n are

$$q_{n+1}(u) = q_n(u) + \frac{u^2}{4n^2-1} q_{n-1}(u), \ n \geq 1, \tag{2}$$

$$q_n'(u) = q_n(u) - \frac{u}{2n-1} q_{n-1}(u), \ n \geq 1. \tag{3}$$

Using hypergeometric functions, we have $q_n(u) = {}_1F_1(-n; -2n; 2u)$. They are normalized according to $q_n(0) = 1$, and thus differ from the

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monic polynomials $\theta_n(u)$ in Grosswald's monograph [4]:

$$\theta_n(u) = \frac{(2n)!}{n!2^n} q_n(u).$$

The polynomials θ_n are sometimes called the reverse Bessel polynomials and $y_n(u) = u^n\theta_n(\frac{1}{u})$ the ordinary Bessel polynomials. These Bessel polynomials are, then, written as

$$y_n(u) = \frac{(2n)!}{n!2^n} u^n q_n(\frac{1}{u}) = \sum_{k=0}^{n} \frac{(n+k)!}{2^k k!(n-k)!} u^k. \tag{4}$$

The *linearization problem* is the problem of finding the coefficients $\beta_k^{(n,m)}(a_1, a_2)$ in the expansion of the product $P_n(a_1u)Q_m(a_2u)$ of two polynomials systems in terms of a third sequence of polynomials $R_k(u)$,

$$P_n(a_1u)Q_m(a_2u) = \sum_{k=0}^{n+m} \beta_k^{(n,m)}(a_1, a_2) R_k(u). \tag{5}$$

The polynomials P_n, Q_m and R_k belong to three different polynomial families. In the case $P = Q = R$ and $a_1 = a_2 = 1$, we get the (standard) linearization or Clebsch-Gordan-type problem. If $Q_m(u) \equiv 1$, we are faced with the so-called connection problem.

In the case $P = Q = R$ and $a_1 = a,\ a_2 = 1 - a$, we get the Berg-Vignat linearization problem. And, finally, in the case $P = Q = R$ and for any a_1, a_2, we get a new linearization problem.

In this paper, we are interested by this new linearization problem and by the linearization coefficients $\beta_k^{(n,m)}(a_1, a_2)$ in the case of the Bessel polynomials which are defined by

$$q_n(a_1u)q_m(a_2u) = \sum_{k=0}^{n+m} \beta_k^{(n,m)}(a_1, a_2) q_k(u). \tag{6}$$

For example, we have

$$q_3(a_1u)q_5(a_2u) = \sum_{k=0}^{8} \beta_k^{(3,5)}(a_1, a_2) q_k(u) \tag{7}$$

where

$$\beta_8^{(3,5)}(a_1,a_2) = 143\,{a_1}^3a_2^5,$$

$$\beta_7^{(3,5)}(a_1,a_2) = -\frac{143}{5}\,{a_1}^2a_2^4\left(12\,a_1a_2 - 5\,a_1 - 2a_2\right),$$

$$\beta_6^{(3,5)}(a_1,a_2) = \frac{11}{5}\,a_1\,{a_2}^3\left(-140\,{a_1}^2\,a_2 + 35\,{a_1}^2 + 30\,a_1\,a_2 + 5\,{a_2}^2 - 56\,a_1\,{a_2}^2 + 126\,{a_1}^2\,{a_2}^2\right),$$

$$\beta_5^{(3,5)}(a_1,a_2) = {a_2}^2({a_2}^3 + 42\,{a_1}^2a_2 + 28\,{a_1}^3 + 84\,{a_1}^2\,{a_2}^3 - 84\,{a_1}^3\,{a_2}^3 + 210a_1^3a_2^2 - 21a_1a_2^3 - 147a_1^3a_2 + 15a_1a_2^2 - 126a_1^2a_2^2),$$

$$\beta_4^{(3,5)}(a_1,a_2) = \frac{1}{3}\,a_2(245a_1^3a_2^2 + 21a_1^3a_2^4 - 140a_1^3a_2 - 140a_1^3a_2^3 + 35a_1a_2^4 - 75a_1a_2^3 - 56a_1^2a_2^4 + 210a_1^2a_2^3 - 210a_1^2a_2^2 + 5a_2^3 + 56a_1^2a_2 + 21a_1^3 - 5a_2^4 + 35a_1a_2),$$

$$\beta_3^{(3,5)}(a_1,a_2) = a_1^3 + \frac{5}{3}a_1^3a_2^4 - \frac{5}{3}a_1a_2^5 - \frac{35}{3}a_1^3a_2^3 + \frac{5}{3}a_2^3 - \frac{50}{3}a_1a_2^3 + \frac{75}{7}a_1a_2^4 + \frac{20}{3}a_1a_2^2 - \frac{50}{21}a_2^4 - 10a_1^2a_2^4 + 6a_1^2a_2 + 30a_1^2a_2^3 - \frac{80}{3}a_1^2a_2^2 + 20a_1^3a_2^2 - 10a_1^3a_2 + \frac{5}{7}a_2^5 + \frac{2}{3}a_1^2a_2^5,$$

$$\beta_2^{(3,5)}(a_1,a_2) = -\frac{1}{105}(a_1+a_2-1)(140a_1^2a_2^2 + 126a_1^2 - 315a_1^2a_2 - 385a_1a_2^2 + 315a_1a_2 + 70a_1a_2^3 - 70a_2^3 + 140a_2^2 + 5a_2^4),$$

$$\beta_1^{(3,5)}(a_1,a_2) = \frac{1}{15}(a_1+a_2-1)(3a_1^2 + 15a_1a_2 - 15a_1 - 15a_2 + 5a_2^2),$$

$$\beta_0^{(3,5)}(a_1,a_2) = -a_1 - a_2 + 1.$$

For $n,m \geq 1$ and $a_1 = a,\ a_2 = 1-a$, Berg and Vignat [2] have proved the following recurrence relation for $\beta_k^{(n,m)}(a,1-a)$ which they denoted by $\beta_k^{(n,m)}(a)$ [2, Lemma 3.6]:

$$\frac{1}{2k+1}\beta_{k+1}^{(n,m)}(a) = \frac{a^2}{2n-1}\beta_k^{(n-1,m)}(a) + \frac{(1-a)^2}{2m-1}\beta_k^{(n,m-1)}(a), \tag{8}$$

for $k = 0,1,...,m+n-1$. From (8) they derived the positivity of $\beta_k^{(n,m)}(a)$ when $0 \leq a \leq 1$ and also that $\beta_k^{(n,m)}(a) = 0$ for $k < \min(m,n)$. Recently, with J. Zeng [1], we improved this result by giving the explicit single-sum formula for $\beta_k^{(n,m)}(a)$ which was missing in their paper [2].

In this paper, our main result is twofold:

- for any a_1, a_2, a recurrence relation for $\beta_k^{(n,m)}(a_1,a_2)$ is given. This

recurrence relation reduces to the recurrence system (8) when $a_1 = a$ and $a_2 = 1 - a$.
- for any a_1, a_2, an explicit single sum formula for $\beta_k^{(n,m)}(a_1, a_2)$, which provides actually the unique solution of the recurrence relation and, then, becomes a generalization of $\beta_k^{(n,m)}(a)$ given by Atia and Zeng in [1] when $a_1 = a$ and $a_2 = 1 - a$.

Lemma 1. *For $n, m \geq 1$, the recurrence relation fulfilled by $\beta_k^{(n,m)}(a_1, a_2)$, $0 \leq k \leq n + m$ is given by*

$$\beta_{n+m}^{(n+1,m-1)}(a_1, a_2) - \frac{a_1^2(2m-1)(2m+1)}{a_2^2(2n-1)(2n+1)}\beta_{n+m}^{(n-1,m+1)}(a_1, a_2) = 0, \quad (9)$$

and for $0 \leq k \leq n + m - 1$, we have

$$\begin{aligned} &\beta_k^{(n+1,m-1)}(a_1, a_2) - \frac{a_1^2(2m-1)(2m+1)}{a_2^2(2n-1)(2n+1)}\beta_k^{(n-1,m+1)}(a_1, a_2) \\ &= \beta_k^{(n,m-1)}(a_1, a_2) - \frac{a_1^2(2m-1)(2m+1)}{a_2^2(2n-1)(2n+1)}\beta_k^{(n-1,m)}(a_1, a_2). \end{aligned} \quad (10)$$

Proof. In one hand we have

$$q_{n+1}(a_1u)q_{m-1}(a_2u) = \sum_{k=0}^{n+m} \beta_k^{(n+1,m-1)}(a_1, a_2)q_k(u),$$

in the other hand, using (2), we have,

$$\begin{aligned} q_{n+1}(a_1u)q_{m-1}(a_2u) &= \left(q_n(a_1u) + \frac{a_1^2u^2}{(2n-1)(2n+1)}q_{n-1}(a_1u)\right)q_{m-1}(a_2u) \\ &= q_n(a_1u)q_{m-1}(a_2u) + \frac{a_1^2u^2}{(2n-1)(2n+1)}q_{n-1}(a_1u)q_{m-1}(a_2u) \\ &= q_n(a_1u)q_{m-1}(a_2u) + \frac{a_1^2(2m-1)(2m+1)}{a_2^2(2n-1)(2n+1)}q_{n-1}(a_1u)\frac{a_2^2u^2}{(2m-1)(2m+1)}q_{m-1}(a_2u) \\ &= q_n(a_1u)q_{m-1}(a_2u) + \frac{a_1^2(2m-1)(2m+1)}{a_2^2(2n-1)(2n+1)}q_{n-1}(a_1u)\left(q_{m+1}(a_2u) - q_m(a_2u)\right) \end{aligned}$$

where we used again (2), finally, we obtain

$$\begin{aligned} &q_{n+1}(a_1u)q_{m-1}(a_2u) - \frac{a_1^2(2m-1)(2m+1)}{a_2^2(2n-1)(2n+1)}q_{n-1}(a_1u)q_{m+1}(a_2u) \\ &= q_n(a_1u)q_{m-1}(a_2u) - \frac{a_1^2(2m-1)(2m+1)}{a_2^2(2n-1)(2n+1)}q_{n-1}(a_1u)q_m(a_2u), \end{aligned}$$

and because of the degree of polynomials $q_k(u)$ we have (9) and for $0 \leq k \leq n + m - 1$ we have (10).

Theorem 2. *For $i = 0, 1, ..., n + m$, we have*

$$\beta_k^{(n,m)}(a_1, a_2) = \frac{a_1^{-m+k} a_2^{-n+k} (1/2)_k}{4^{m+n-k}(m+n-k)!(1/2)_n(1/2)_m}$$
$$\sum_{i=0}^{m+n-k} a_1^{m+n-k-i} \binom{m+n-k}{i} (n+1-i)_{2i}$$
$$\sum_{j=0}^{m+n-k-i} (-1)^j \binom{m+n-k-i}{j} (-n+k+j+i+1)_{2(m+n-k-i-j)} (k+2-j)_{2j} a_2^{j+i}. \tag{11}$$

which we write using ${}_3F_2$ hypergeometric functions as

Theorem 3. *For $i = 0, 1, ..., n + m$, we have*

$$\begin{aligned}\beta_k^{(n,m)}(a_1, a_2) \quad = \quad & \frac{a_1^{-m+k} a_2^{m} (1/2)_k}{4^{m+n-k}(m+n-k)!(1/2)_n(1/2)_m} \\ & \sum_{i=0}^{m+n-k} a_1^{i} \binom{m+n-k}{m+n-k-i} (-m+k+i+1)_{2(m+n-k-i)} (m-i+1)_{2i} \\ & {}_3F_2\left(\begin{matrix} k+2, \; -k-1, \; -i \\ -m-i, \; m-i+1 \end{matrix} ; a_2 \right) a_2^{-i}.\end{aligned} \tag{12}$$

Remarks.
1. This formula was deduced using the same approach done in [1] pages 4 and 5 by, just, changing a by a_1 and $1 - a$ by a_2.
2. To compute this formula with, for example, Maple, one should compute $\beta_{n+m}^{(n,m)}(a_1, a_2)$, $\beta_{n+m-1}^{(n,m)}(a_1, a_2)$,..., $\beta_0^{(n,m)}(a_1, a_2)$ and then replace n, m by their values (please see the Maple program given in the end of this paper).

Proof of theorem 3. Let us, first, prove that (12) fulfils (9):

$$\beta_{n+m}^{(n+1,m-1)}(a_1, a_2) = \frac{a_1^{n+1} a_2^{m-1} \sqrt{\pi}\,\Gamma(1/2+n+m)}{\Gamma(n+3/2)\Gamma(m-1/2)},$$

and

$$\beta_{n+m}^{(n-1,m+1)}(a_1, a_2) = \frac{a_1^{n-1} a_2^{m+1} \sqrt{\pi}\,\Gamma(1/2+n+m)}{\Gamma(n-1/2)\Gamma(m+3/2)},$$

then

$$\frac{a_1^2(2m-1)(2m+1)}{a_2^2(2n-1)(2n+1)} \beta_{n+m}^{(n-1,m+1)}(a_1, a_2) = \frac{a_1^{n+1} a_2^{m-1} \sqrt{\pi}\,\Gamma(1/2+n+m)}{\Gamma(n+3/2)\Gamma(m-1/2)}.$$

Second, we prove that (12) fulfils (10), so let us substract the rhs from lhs of (10) to obtain

$$\beta_k^{(n+1,m-1)}(a_1,a_2) - \frac{a_1^2(2m-1)(2m+1)}{a_2^2(2n-1)(2n+1)}\beta_k^{(n-1,m+1)}(a_1,a_2)$$

$$-\beta_k^{(n,m-1)}(a_1,a_2) + \frac{a_1^2(2m-1)(2m+1)}{a_2^2(2n-1)(2n+1)}\beta_k^{(n-1,m)}(a_1,a_2)$$

$$= \frac{a_1^{-m+1+k}a_2^{m-1}\sqrt{\pi}\Gamma(1/2+k)}{4^{m+n-k}(m+n-k)!\Gamma(n+3/2)\Gamma(m-1/2)}\times$$

$$\Bigg(\sum_{i=0}^{m+n-k} a_1^i \tbinom{m+n-k}{m+n-k-i}(-m+k+i+2)_{2(n+m-k-i)}(m-i)_{2i}$$

$${}_3F_2([-i,k+2,-k-1],[-m+1-i,m-i],a_2)a_2^{-i}$$

$$-\sum_{i=0}^{m+n-k} a_1^i \tbinom{m+n-k}{m+n-k-i}(-m+k+i)_{2(n+m-k-i)}(m-i+2)_{2i}$$

$${}_3F_2([-i,k+2,-k-1],[-m-i-1,m-i+2],a_2)a_2^{-i}\Bigg)$$

$$-\frac{a_1^{-m+1+k}a_2^{m-1}\sqrt{\pi}\Gamma(1/2+k)}{4^{m+n-k-1}(m+n-k-1)!\Gamma(n+1/2)\Gamma(m-1/2)}\times$$

$$\Bigg(\sum_{i=0}^{m+n-k-1} a_1^i \tbinom{m+n-k-1}{m+n-k-i-1}(-m+k+i+2)_{2(n+m-k-i-1)}(m-i)_{2i}$$

$${}_3F_2([-i,k+2,-k-1],[-m-i+1,m-i],a_2)a_2^{-i}$$

$$-\frac{(m+1/2)a_1}{(n+1/2)a_2}\sum_{i=0}^{m+n-k-1} a_1^i \tbinom{m+n-k-1}{m+n-k-i-1}(-m+k+i+1)_{2(n+m-k-i-1)}(m-i+1)_{2i}$$

$${}_3F_2([-i,k+2,-k-1],[-m-i,m-i+1],a_2)a_2^{-i}\Bigg),$$

because

$$\frac{(2m-1)(2m+1)}{(2n-1)(2n+1)\Gamma(n-1/2)\Gamma(m+3/2)} = \frac{1}{\Gamma(n+3/2)\Gamma(m-1/2)}.$$

Cancelling the common factor

$$\frac{a_1^{-m+1+k}a_2^{m-1}\sqrt{\pi}\Gamma(1/2+k)}{4^{m+n-k-1}(m+n-k-1)!\Gamma(n+1/2)\Gamma(m-1/2)}$$

in both quantities, we get

$$\frac{1}{4(m+n-k)(n+1/2)}\times$$

$$\Bigg(\sum_{i=0}^{m+n-k} a_1^i \tbinom{m+n-k}{m+n-k-i}(-m+k+i+2)_{2(n+m-k-i)}(m-i)_{2i}$$

$${}_3F_2([-i,k+2,-k-1],[-m+1-i,m-i],a_2)a_2^{-i}$$

$$-\sum_{i=0}^{m+n-k} a_1^i \tbinom{m+n-k}{m+n-k-i}(-m+k+i)_{2(n+m-k-i)}(m-i+2)_{2i}$$

$${}_3F_2([-i,k+2,-k-1],[-m-i-1,m-i+2],a_2)a_2^{-i}\Bigg)$$

$$-\Bigg(\sum_{i=0}^{m+n-k-1} a_1^i \tbinom{m+n-k-1}{m+n-k-i-1}(-m+k+i+2)_{2(n+m-k-i-1)}(m-i)_{2i}$$

$${}_3F_2([-i,k+2,-k-1],[-m-i+1,m-i],a_2)a_2^{-i}$$

$$-\frac{(m+1/2)a_1}{(n+1/2)a_2}\sum_{i=0}^{m+n-k-1} a_1^i \tbinom{m+n-k-1}{m+n-k-i-1}(-m+k+i+1)_{2(n+m-k-i-1)}(m-i+1)_{2i}$$

$${}_3F_2([-i,k+2,-k-1],[-m-i,m-i+1],a_2)a_2^{-i}\Bigg),$$

equivalently

$$\frac{1}{4(m+n-k)(n+1/2)}\times$$

$$\Bigg(\sum_{i=0}^{m+n-k} a_1^i \tbinom{m+n-k}{m+n-k-i}(-m+k+i+2)_{2(n+m-k-i)}(m-i)_{2i}$$

$${}_3F_2([-i,k+2,-k-1],[-m+1-i,m-i],a_2)a_2^{-i}$$

$$-\sum_{i=0}^{m+n-k} a_1^i \tbinom{m+n-k}{m+n-k-i}(-m+k+i)_{2(n+m-k-i)}(m-i+2)_{2i}$$

$${}_3F_2([-i,k+2,-k-1],[-m-i-1,m-i+2],a_2)a_2^{-i}\Bigg)$$

$$-\sum_{i=0}^{m+n-k-1} a_1^i \tbinom{m+n-k-1}{m+n-k-i-1}(-m+k+i+2)_{2(n+m-k-i-1)}(m-i)_{2i}$$

$${}_3F_2([-i,k+2,-k-1],[-m-i+1,m-i],a_2)a_2^{-i}$$

$$+\frac{(m+1/2)a_1}{(n+1/2)a_2}\sum_{i=0}^{m+n-k-1} a_1^i\binom{m+n-k-1}{m+n-k-i-1}(-m+k+i+1)_{2(n+m-k-i-1)}(m-i+1)_{2i}$$

$$_3F_2([-i,k+2,-k-1],[-m-i,m-i+1],a_2)a_2^{-i}.$$

To prove that this expression vanishes, it suffices to prove that the coefficient of a_1^j vanishes. The coefficient of a_1^j is given by

$$\frac{1}{4(m+n-k)(n+1/2)}\times$$

$$\Big(a_1^j\binom{m+n-k}{m+n-k-j}(-m+k+j+2)_{2(n+m-k-j)}(m-j)_{2j}$$

$$_3F_2([-j,k+2,-k-1],[-m+1-j,m-j],a_2)a_2^{-j}$$

$$-a_1^j\binom{m+n-k}{m+n-k-j}(-m+k+j)_{2(n+m-k-j)}(m-j+2)_{2j}$$

$$_3F_2([-j,k+2,-k-1],[-m-j-1,m-j+2],a_2)a_2^{-j}\Big)$$

$$-a_1^j\binom{m+n-k-1}{m+n-k-j-1}(-m+k+j+2)_{2(n+m-k-j-1)}(m-j)_{2j}$$

$$_3F_2([-j,k+2,-k-1],[-m-j+1,m-j],a_2)a_2^{-j}$$

$$+\frac{(m+1/2)a_1}{(n+1/2)a_2}a_1^{j-1}\binom{m+n-k-1}{m+n-k-j}(-m+k+j)_{2(n+m-k-j)}(m-j+2)_{2(j-1)}$$

$$_3F_2([-(j-1),k+2,-k-1],[-m-(j-1),m-(j-1)+1],a_2)a_2^{-(j-1)}.$$

A short computation (with Maple) of this quantity gives zero:
$Q1 := ((a1^i * binomial(n+m-k, n+m-k-i) * pochhammer(-m+k+i+2, 2*n+2*m-2*k-2*i) * pochhammer(m-i, 2*i) * hypergeom([-i, k+2, -k-1], [-m-i+1, m-i], a2) * a2^{(-i)}) - (a1^i * binomial(n+m-k, n+m-k-i) * pochhammer(-m+k+i, 2*n+2*m-2*k-2*i) * pochhammer(m-i+2, 2*i) * hypergeom([-i, k+2, -k-1], [m-i+2, -m-i-1], a2) * a2^{(-i)})) - 4*(n+m-k)*(n+1/2)*(a1^i * binomial(n+m-k-1, n+m-k-i-1) * pochhammer(-m+k+i+2, 2*n+2*m-2*k-2*i-2) * pochhammer(m-i, 2*i) * hypergeom([-i, k+2, -k-1], [-m-i+1, m-i], a2) * a2^{(-i)}); Q2 := -4*(n+m-k)*(m+1/2)*a1/a2*(a1^{(i-1)} * binomial(n+m-k-1, n+m-k-(i-1)-1) * pochhammer(-m+k+(i-1)+1, 2*n+2*m-2*k-2*(i-1)-2) * pochhammer(m-(i-1)+1, 2*(i-1)) * hypergeom([-(i-1), k+2, -k-1], [-m-(i-1), m-(i-1)+1], a2) * a2^{(-i+1)}); simplify(Q1-Q2);$

Particular case.
Let us prove that (10) reduces to (8) when $a_1 = a,\ a_2 = 1-a$.
From (10) we have

$$\frac{(1-a)^2}{2m-1}\beta_k^{(n,m-1)}(a,1-a) - \frac{a^2(2m+1)}{(4n^2-1)}\beta_k^{(n-1,m)}(a,1-a)$$
$$= \frac{(1-a)^2}{2m-1}\beta_k^{(n+1,m-1)}(a,1-a) - \frac{a^2(2m+1)}{(4n^2-1)}\beta_k^{(n-1,m+1)}(a,1-a). \quad (13)$$

equivalently

$$\frac{(1-a)^2}{2m-1}\beta_k^{(n,m-1)}(a,1-a) = \frac{a^2(2m+1)}{(4n^2-1)}\beta_k^{(n-1,m)}(a,1-a)$$
$$+\frac{(1-a)^2}{2m-1}\beta_k^{(n+1,m-1)}(a,1-a) - \frac{a^2(2m+1)}{(4n^2-1)}\beta_k^{(n-1,m+1)}(a,1-a). \quad (14)$$

Adding $\frac{a^2}{2n-1}\beta_k^{(n-1,m)}(a,1-a)$ to both sides, we get

$$\frac{(1-a)^2}{2m-1}\beta_k^{(n,m-1)}(a,1-a) + \frac{a^2}{2n-1}\beta_k^{(n-1,m)}(a,1-a)$$
$$= \frac{a^2(2m+1)}{(4n^2-1)}\beta_k^{(n-1,m)}(a,1-a) + \frac{(1-a)^2}{2m-1}\beta_k^{(n+1,m-1)}(a,1-a)$$
$$-\frac{a^2(2m+1)}{(4n^2-1)}\beta_k^{(n-1,m+1)}(a,1-a) + \frac{a^2}{2n-1}\beta_k^{(n-1,m)}(a,1-a). \quad (15)$$

According to (8) le lhs is equal to $\frac{1}{2k+1}\beta_{k+1}^{(n,m)}(a,1-a)$.
Using (6), the rhs becomes

$$-\frac{a^2(2m+1)}{(4n^2-1)}q_{n-1}(au)q_{m+1}((1-a)u) + \frac{a^2}{2n-1}(1+\frac{2m+1}{2n+1})q_{n-1}(au)q_m((1-a)u)$$
$$+\frac{(1-a)^2}{2m-1}q_{n+1}(au)q_{m-1}((1-a)u).$$

Using (2), we obtain

$$-\frac{a^2(2m+1)}{(4n^2-1)}q_{n-1}(au)\left(q_m((1-a)u) + \frac{(1-a)^2u^2}{4m^2-1}q_{m-1}((1-a)u)\right)$$
$$+\frac{a^2}{2n-1}(1+\frac{2m+1}{2n+1})q_{n-1}(au)q_m((1-a)u)$$
$$+\frac{(1-a)^2}{2m-1}\left(q_n(au) + \frac{a^2u^2}{4n^2-1}q_{n-1}(au)\right)q_{m-1}((1-a)u).$$

After simplification, we get the rhs of (8).

2. Applications

1- These coefficients $\beta_k^{(n,m)}(a_1,a_2)$ with $a_1+a_2 \neq 1$ can be applied in: if X and Y are two student random variables with n and m degrees of freedom then the linear combination a_1X+a_2Y has for characteristic function

$$e^{(-a_1u-a_2u)}q_n(a_1u)q_m(a_2u) = e^{(-a_1u-a_2u)}\sum_{k=0}^{n+m}\beta_k^{n,m}(a_1,a_2)q_k(u),$$

On the other hand, we have

$$a_1X+a_2Y = (a_1+a_2)(\frac{a_1}{(a_1+a_2)}X+\frac{a_2}{(a_1+a_2)}Y) = (a_1+a_2)(\tilde{a_1}X+\tilde{a_2}Y)$$

with $\tilde{a_1}+\tilde{a_2}=1$ then it exists a NON TRIVIAL relation between the coefficients $\beta_k^{(n,m)}(a_1,a_2)$ and the coefficients $\beta(a_1,1-a_1)$ which is not clear in their expressions.
2- For $a_1 = a,\ a_2 = 1-a$, these coefficients $\beta_k^{(n,m)}(a,1-a)$ give a formula reducing a sum of hypergeometric functions ${}_3F_2$ to ${}_2F_1$:

Theorem 4. *Taking into account* (12) *and formulas* $(7)-(8)$ *given in* [1], *we get: for* $k \geq \lceil (n+m-1)/2 \rceil$

$$\frac{a^{2n+2m-2k}(1-a)^{-m-n+k}\Gamma(n+m+2)}{\Gamma(-n-m+2k+2)}\,{}_2F_1\left(\begin{matrix}-m+k+1,\ -2n-2m+2k\\ -n-m+2k+2\end{matrix};\frac{1}{a}\right)$$

$$= \sum_{i=0}^{m+n-k} a^i\binom{m+n-k}{m+n-k-i}(-m+k+i+1)_{2(m+n-k-i)}(m-i+1)_{2i}$$

$${}_3F_2\left(\begin{matrix}k+2,\ -k-1,\ -i\\ -m-i,\ m-i+1\end{matrix};1-a\right)(1-a)^{-i}$$

and for $k \leq \lfloor (n+m-1)/2 \rfloor$

$$\frac{(-a)^{n+1+m}(1-a)^{-m-n+k}\Gamma(2n+2m-2k+1)\Gamma(n-k)}{\Gamma(n+m-2k)\Gamma(-m+k+1)}\,{}_2F_1\left(\begin{matrix}n-k,\ -n-m-1\\ n+m-2k\end{matrix};\frac{1}{a}\right)$$

$$= \sum_{i=0}^{m+n-k} a^i\binom{m+n-k}{m+n-k-i}(-m+k+i+1)_{2(m+n-k-i)}(m-i+1)_{2i}$$

$${}_3F_2\left(\begin{matrix}k+2,\ -k-1,\ -i\\ -m-i,\ m-i+1\end{matrix};1-a\right)(1-a)^{-i}$$

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Please find next a Maple program which, not only, tests that our formula is right from $min(n,m)$ to $n+m$ but, also, show that $\beta_k^{n,m}(a,1-a)=0$ for $k<min(n,m)$.

$> restart;$

$> A := (n,m)- > q(n,a1*u)*q(m,a2*u) - sum(beta(n,m,k,a1,a2) * q(k,u), k = min(n,m)..n+m :)$

We assume n less or equal m. This program runs from $min(n,m)$ untill $n+m$, take any values of $n,\ m$, for example 2 and 8

$> AA := A(2,8) :;$

$> alpha := (n,k)- > n! * (2*n-k)! * 2^k/(2*n)!/(n-k)!/k! :$

$> q := (n,u)- > sum(alpha(n,k)*u^k, k = 0..n) :;$

$$> beta := \quad (n,m,k,a1,a2)- > factor(a1^{(-m+k)} * a2^m * Pi^{(1/2)} \\ *GAMMA(1/2+k) * sum(a1^i * binomial(n+m-k, n+m-k-i) \\ *pochhammer(-m+k+i+1, 2*n+2*m-2*k-2*i) \\ *pochhammer(m-i+1, 2*i) * simplify(hypergeomt(n,m,i,k)) \\ *a2^{(-i)}, i = 0..n+m-k)/(4^(n+m-k))/(n+m-k)! \\ /GAMMA(n+1/2)/GAMMA(m+1/2)) :;$$

$> AAA := factor(AA) :;$

$> hypergeomt := (n,m,i,k)- > simplify(hypergeom([-i,k+2,-k-1], [-m-i, m-i+1], a2)) :$

$> collect(factor(simplify(AAA)), u);$

$$1/5*(-1+a1+a2)*(5*a2*a1-5*a1-5*a2+2*a2^2)*u \\ +1/5*(-1+a1+a2)*(5*a2*a1-5*a1-5*a2+2*a2^2-5);$$

We meet again that $beta(n,m,k,a,1-a)$ vanish for $k<min(n,m)$.